MEASURING THE FRACTAL DIMENSIONS OF EMPIRICAL CARTOGRAPHIC CURVES

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Abstract

The fractal dimension of a curve is a measure of its geometric complexity and can be any non-integer value between 1 and 2 depending upon the curve's level of complexity. This paper discusses an algorithm, which simulates walking a pair of dividers along a curve, used to calculate the fractal dimensions of curves. It also discusses the choice of chord length and the number of solution steps used in computing fracticality. Results demonstrate the algorithm to be stable and that a curve's fractal dimension can be closely approximated. Potential applications for this technique include a new means for curvilinear data compression, description of planimetric feature boundary texture for improved realism in scene generation and possible two-dimensional extension for description of surface feature textures.

INTRODUCTION

The problem of describing the forms of curves has vexed researchers over the years. For example, a coastline is neither straight, nor circular, nor elliptic and therefore Euclidean lines cannot adquately describe most real world linear features. Imagine attempting to describe the boundaries of clouds or outlines of complicated coastlines in terms of classical geometry. An intriguing concept proposed by Mandelbrot (1967, 1977) is to use fractals to fill the void caused by the absence of suitable geometric representations. A fractal characterizes curves and surfaces in terms of their complexity by treating dimension as a continuum. Normally, dimension is an integer number (1 for curves, 2 for areas, and 3 for volumes); however, fractal dimensions may vary anywhere between 1 and 2 for a curve and 2 and 3 for a surface depending upon the irregularity of the form. Although individual fractals have been around since the 1900's,

Mandelbrot was the first to recognize their applications outside of mathematics.

This paper discusses an algorithm, written in an interactive setting, designed to measure the fracticality of a curve and additions to theory. It also presents results from examining several cartographic curves.

DEFINITION OF FRACTALS AND SELF-SIMILARITY

In Euclidean geometry every curve has a dimension of 1 and every plane has a dimension of 2. This is generally referred to as the topological dimension (Dt). These dimensions remain constant no matter how complex or irregular a curve or plane may be. For example, the west coast of Great Britain contains many irregularities, but the topological dimension remains 1.

In the fractal domain a curve's dimension may be between 1 and 2 according to its complexity. The more contorted a straight line becomes, the higher its fractal dimension. Similarly, a plane's dimension may be a non-integer value between 2 and 3. The fractal dimension for any curve or surface is denoted by (D) and within this framework: $D \geq Dt$. Mandelbrot (1977) proposes the following definition for a fractal: "A fractal will be defined as a set for which the Hausdorff-Besicovitch dimension strictly exceeds the topological dimension." The precise definition of the Hausdorff-Besicovitch dimension can be found in Besicovitch and Ursell (1937).

Central to the concept of fractals is the notion of self-similarity. Self-similarity means that for any curve or surface a portion of the curve or surface can be considered a reduced image of the whole. However, seldom in nature (crystals are one exception) does self-similarity occur and therefore a statistical form of self-similarity is often encountered. In other words, if a curve or surface is examined at any scale it will resemble the whole in a statistical sense; therfore, D will remain constant. Brownian motion is an excellent example of statistical self-similarity. Because of this principle, a curve can be decomposed into $N=r$ nonoverlapping parts and each subsegment has a length of $1/r=1/N$. Similarly, a unit square can be divided into $N=r^2$ squares, where the similarity ratio is $r(N) = 1/r = 1/N^{1/2}$. In either case the following equation applies:

$$D = \log N/\log (1/r) \qquad (1)$$

and could be called the shape's similarity dimension. D can also be expressed as:

$$D = \log (N/N_o)/\log (\lambda_o/\lambda) \qquad (2)$$

where λ_o and λ are two sampling intervals and N_o and N are the number of such intervals contained. If a curve resembles a straight line then when the sampling interval is halved, N doubles and the proportion equals 1. The majority of cartographic curves are not straight lines and therefore N will more than double causing D to be greater than 1. The principle of self-similarity is dismissed by Goodchild (1980), Hakanson (1978), and Scheidegger (1970). Hakanson, for example, points out the absurdity of postulating the validity of self-similarity down to the size of the pebbles on the coastline and at the molecular interstices of those pebbles. Goodchild demonstrates that although Richardson (1961) found the west coast of Britian to have a constant D of 1.25 over sampling intervals between 10 and 1000km., he found the east coast to vary between 1.15 and 1.31 for a similar sampling interval. This suggests that whatever created the irregularities on the coastline acted at specific scales. Goodchild states that since self-similarity is only one aspect of the fractal approach, it would be unwise to reject the entire concept.

DEVELOPMENT OF THE FRACTAL CURVE ALGORITHM AND EXTENSION OF THEORY

The following original algorithm is based on the earlier empirical work performed by Richardson (1961) and later extended by Mandelbrot (1967). Richardson measured the lengths of several frontiers by manually walking a pair of dividers along the outline so as to count the number of steps. The opening of the dividers (n) was fixed in advance and a fractional side was estimated at the end of the walk. The main purpose in this section of Richardson's research was to study the broad variation of Σn with n.

Richardson produced a scatterplot in which he plotted log total length against log step size for five land frontiers and a circle. Mandelbrot (1967) discovered a relationship between the slope (β) of the lines and fractal dimension (D). To Richardson the slope had no theoritical meaning, but to Mandelbrot it could be used as an estimate of 1-D, which leads to:

$$D = 1 - \beta \tag{3}$$

The algorithm simulates walking a pair of dividers along a curve and counts the number of steps. In cases where more than one intersection occurs, the intersection which comes first in order forward along the curve is selected. To be more accurate, step size (prescribed opening of the dividers) is called chord length (cl) and the number of steps is called the number of chord lengths.

In order to begin walking the dividers along the curve, the dividers must be set to some opening. The curves used in this research are not infinitely subdivided fractal curves so that selection of the initial chord length must be based on some attribute of the curve. For a very contorted curve it would be meaningless to choose a chord length many times shorter than the shortest line segment. If an extremely short chord length is selected, an attempt to examine the fractal character of a curve would extend beyond the primitive subelements used to represent the geometry of the resulting form. In other words, beyond this lower limit of primitive subelements, the curve's fractal dimension behaves as if it is a straight line. A suggested initial chord length is determined by calculating the distance between each two consecutive points on the curve and taking 1/2 the average distance. The average distance is divided by 2 because the sampling theorem states one should sample at 1/2 the wavelength so that no significant variation escapes. This presents an approximate lower limit as to the selection of the initial chord length. Although the accuracy of this method is dependent on the manner in which the curve is digitized, the form of the curve often dictates this manner.

After the initial chord length is determined, the algorithm computes the distance between the first two points on the curve using the standard distance formula. If the distance is greater than chord length (cl), a new point is interpolated between points 1 and 2 using the following interpolation equations:

$$DP = (cl - DIST1) / (DIST1 - DISTA) \tag{4}$$

$$XNEW = X_1 + DP * (X_2 - X_1) \tag{5}$$

$$YNEW = Y_1 + DP * (Y_2 - Y_1) \tag{6}$$

where DP = distance proportion
DIST1 = distance between the present point and the first forward point on the curve
DISTA = distance between the present point and the second forward point on the curve

XNEW = new X- coordinate
YNEW = new Y- coordinate
X,Y = X and Y coordinates of point 1 and 2.

Figure 1 demonstrates how a point is interpolated on a straight line segment.

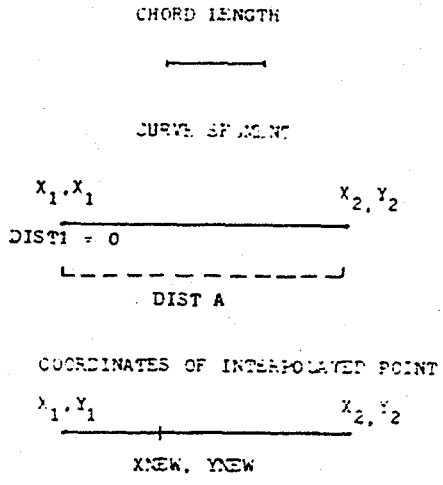

Figure 1. Interpolating on a straight line segment.

If the distance is less than the chord length, the distance between points 1 and 3 (DISTC) is computed. If DISTC is greater than the chord length, it is known that the chord length segment intersects between points 2 and 3 and that the distance between these points is determined (DISTB); See Figure 2a.

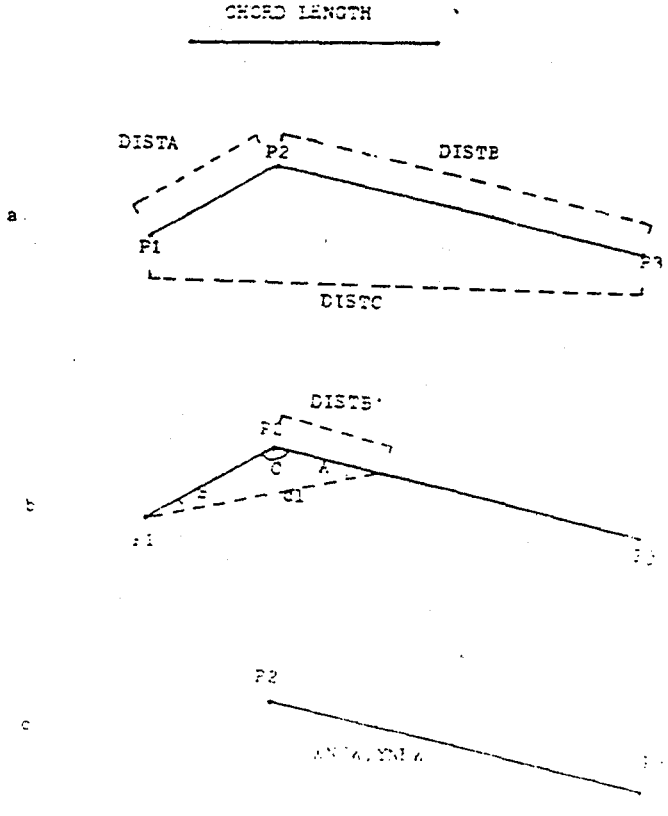

Figure 2. Three point interpolation.

The point of intersection is computed using trigonometric functions. An angle C is determined using the law of cosines.

$$C = \cos^{-1} \frac{DISTB^2 + DISTA^2 - DISTC^2}{2 * DISTB * DISTA} \qquad (7)$$

Since angle C is known, an angle A, which is the angle the chord length intersects between points 2 and 3, can be computed.

$$A = SIN^{-1} ((DISTA * \sin C)/cl) \qquad (8)$$

Now that two angles are known, angle B is easily computed. Because angles A and B are known, a side (DISTB') can be calculated; see Figure 2b.

$$DISTB' = (DISTA * \sin B)/\sin A \qquad (9)$$

DISTB' provides the distance, from point 2, in which the chord length's intersection is located on the segment between points 2 and 3. A distance proportion is calculated using:

$$DP = DISTB'/DISTB \qquad (10)$$

Since the distance proportion and the X,Y coordinates for points 2 and 3 are known, the equations used to interpolate for a straight line segment can be used to determine the new coordinates: see Figure 2c. After the new point is located, this new point becomes point 1 and the next two forward points on the curve become points 2 and 3. Each time a chord length's intersection is determined, 1 is added to the number of chord lengths.

In the case where DISTC is less than the chord length, the third point is incremented by 1 (fourth point) and the distance again checked. This continues until the distance is greater than the chord length or the end of the curve is encountered; see Figure 3.

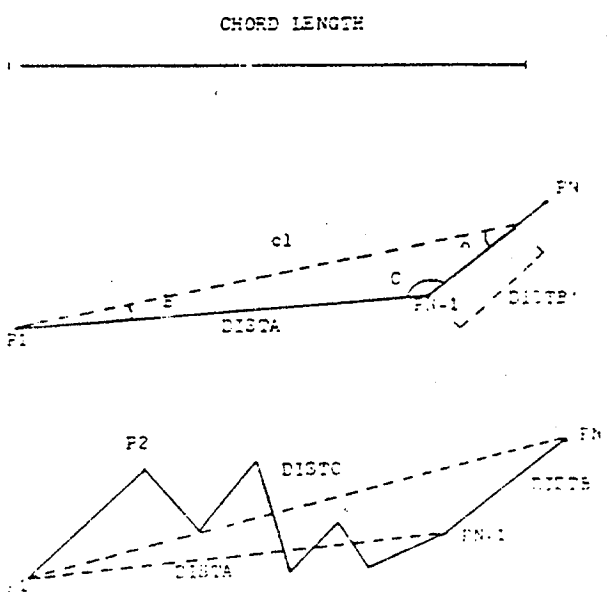

Figure 3. More than three point interpolation.

When the distance does become greater than the chord length, the chord length's point of intersection is determined by using the same trigonometric equations as discussed above. The only difference is the sides of the triangles may be longer. At the end of the curve, if the chord length is greater than DISTA, the portion of the remaining chord length is added to the number of chord lengths.

After the dividers are walked along the curve with the initial chord length, the dividers are opened to another distance. This distance is a geometric adding of the first chord length. For example, if the initial chord length is 2, then the subsequent chord lengths would be 4, 8, 16, 32, 64, and so on. This eliminates biasing when using linear regression because on a logarithmic scale, geometric adding provides equal spacing between the chord lengths.

The number of solution steps or the number of times the dividers, with different chord lengths, are walked along the curve is limited by the number of chord lengths used to estimate length. The minimum number of chord lengths used to approximate length is 5. This is chosen to provide consistency among results as opposed to using a variable limit, but is subject to change pending additional research.

After each time the dividers are walked along the curve, the number of chord lengths and the corresponding chord lengths are saved. These are used in the linear regression where log line length (number of chord lengths *chord length) is regressed against log chord length. A curve's fractal dimension is determined by using equation 3.

To provide an indication of the proportion of variance in the response variable explained by the describing variable, r^2 is computed. This value plays an important role in determining the optimum number of solution steps. A low r^2, for example when the number of solution steps equals 12, means the initial chord length falls below the primitive subelements. A low r^2 is determined by decreasing the number of solution steps and comparing the two values. The desirable number of solution steps is indicated when r^2 reaches its maximum without the number of steps falling below 5. A linear regression with less than 5 points opens up some criticisms as to the validity of results and it should be emphasized the linear regression is used as a parameter estimate and not for statistical inferences.

EXAMPLES AND RESULTS

Of the five land-frontiers Richardson measured, four have been point digitized and used in this study. They are: coast of the Australian mainland; coast of South Africa, starting from Swakopmund to Cape Sta. Lucia; frontier between Spain and Portugal, moving south to north; and the west coast of Great Britain, from Land's End to Duncansby Head. Table 1 shows D as the result of Richardson's measurements and the new D suggested by this research. The expected discrepancy is the result of the digitization process because digitization allows the capture of minute detail in a curve, and since these curves were digitized at a larger scale, a higher D is anticipated.

Curve	Slope (β)	D (1-β)	New D
West Coast of Great Britain	-.25	1.25	1.2671
Coast of Australia	-.13	1.13	1.1490
Coast of South Africa	-.02	1.02	1.0356
Land-frontier between Spain and Portugal	-.14	1.14	1.1014

Table 1. Result from Richardson's (1961) research, corresponding fractal dimension and the new suggested fractal dimension.

For this paper, Kodiak Island is used to demonstrate how the fractal curve algorithm operates. The curve was digitized in trace mode with delta minimum and delta maximum variations at .002 and .05 respectively. The outline contains 1653 points and is in Figure 4.

Figure 4. Kodiak Island with 1653 points where the fractal dimension equals 1.3105.

The results from calculating D are in Table 2 where the different initial chord lengths are selected to show the possible variations in D over a number of sampling intervals. The results show D to vary from 1.1836 to 1.3714. These variations in D reflect a lack of self-similarity in the curve.

Initial Chord Length	D	R-SQ	No. of Solution Steps
.00400	1.1836	.827614	10
* .01833	1.2619	.916728	8
** .03666	1.3025	.954146	7
.05894	1.3105	.976810	6
.07500	1.3466	.964087	6
.08000	1.3660	.972126	6
.10000	1.3714	.975220	5

*Suggested initial chord length
**Average segment length

TABLE 2.
Kodiak Island at 1:1,000,000 with 1653 points

The selection of an extremely short initial chord length of .01833 represents examining the curve below its primitive subelements and biases D toward a straight line. The corresponding scatterplot in Figure 5 displays a curvature of the data points resulting in a lower R-SQ.

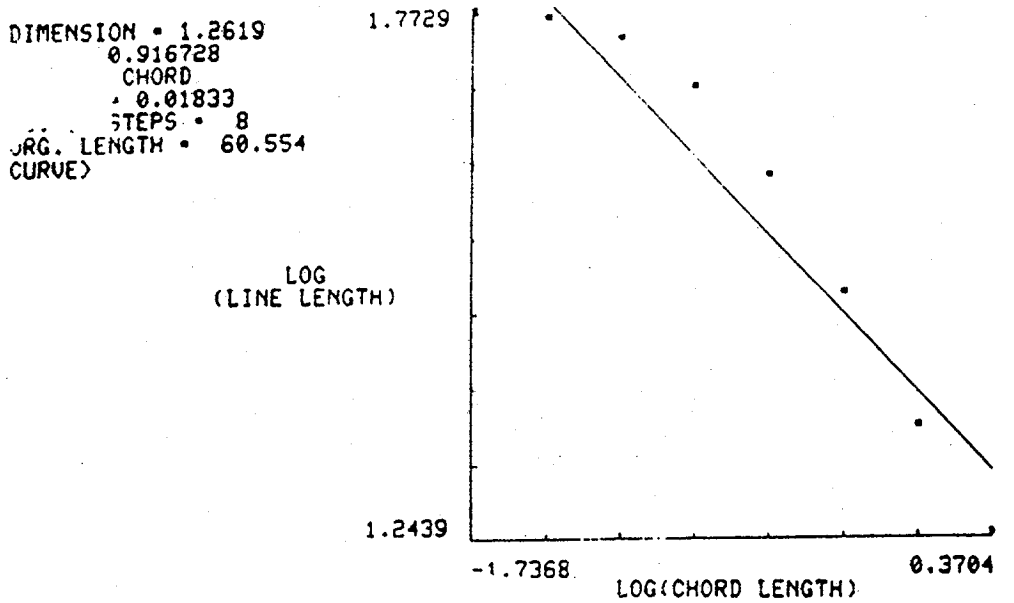

Figure 5. Scatterplot for Kodiak Island (1653 points) where the initial chord length equals .01833.

It is this type of curvature, resembling the shape of a rainbow, that indicates the shortness of the chord length. The chord length of .01833 is the average segment length and is calculated by computing the distance between each two consecutive points on the curve, summing the distances, and dividing by the number of segments. The suggested D to represent the curve is 1.3105. The most appropriate D value is determined from the minimum amount of curvature present in the scatterplot resulting in the relatively high R-SQ value. The suggested initial chord length of .03666 is still too small, indicated by the low R-SQ value, but represents a starting point at which to determine D.

A thinned version of Kodiak Island is in Figure 6 and contains 1000 points.

Figure 6. Kodiak Island with 1000 points where the fractal dimension equals 1.2949.

The elimination of 653 points is accomplished with a program which deletes excessive points within a certain chord length. Table 3 indicates D varying between 1.214 and 1.3659.

Initial Chord Length	D	R-SQ	No. of Solution Steps
.00700	1.1778	.861745	9
.00800	1.2144	.867113	9
* .02947	1.2558	.942722	7
** .05894	1.2949	.974398	6
.10000	1.3659	.971356	5

*Suggested initial chord length
**Average segment length

TABLE 3.
Kodiak Island at 1:1,000,000 with 1000 points

The comparison between the same chord length of .05894 for the original and thinned islands displays how stable the algorithm is to measure D. This initial chord length produced a D of 1.3105 (1653 points) and 1.2949 (1000 points) giving a 1.19% difference. The D for the 1000 point island is expected to be slightly lower because any data thinning process normally removes some complexity from the feature. The proposed D for the thinned island is approximately 1.2949 and the scatterplot is in Figure 7.

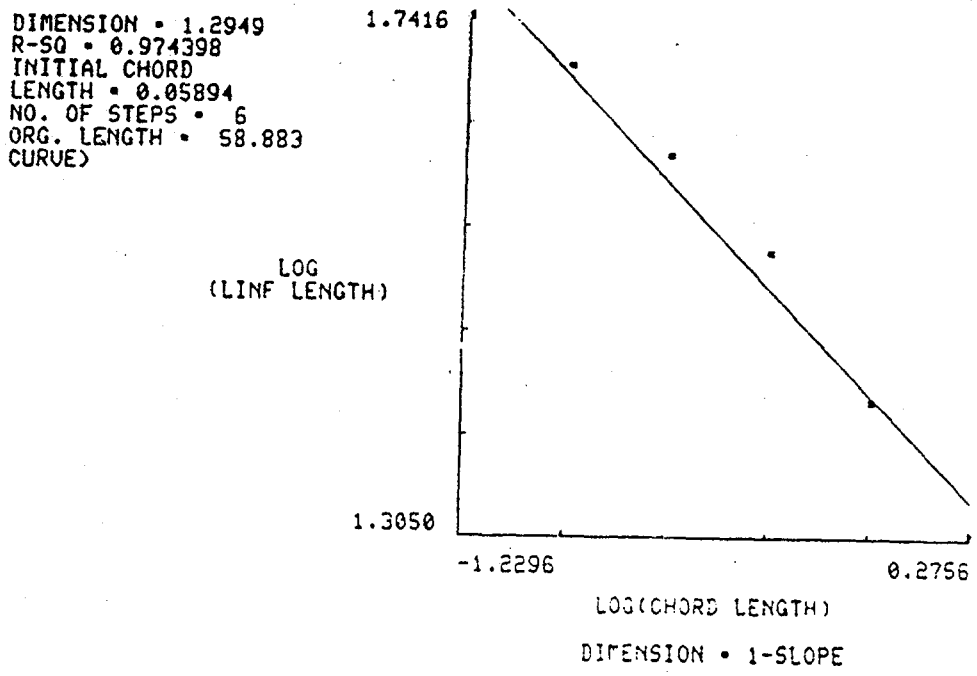

Figure 7. Scatterplot for Kodiak Island (1000 points) where the initial chord length equals .05894.

SUMMARY AND CONCLUSIONS

The results based on the previous empirical curves point out the importance of selecting the appropriate initial chord length. A chord length which is too short

is easily detected by either examining the amount of curvature present in the scatterplot or the low R-SQ value. Normally, the suggested initial chord length falls within this category, but it must be emphasized that this chord length is merely a beginning point. The ideal initial chord length, which produces the most appropriate D, is selected by observing the behavior of the scatterplots, R-SQ values, and the number of solution steps. This research, like Richardson's work, indicates that from 5 to 8 solution steps are sufficient to determine the slope of the regression line and thus fracticality.

The results also indicate the fractal curve algorithm to be stable, and that it is able to closely approximate D. The variations in D, over a number of sampling intervals, reflect a need to examine the effects of self-similarity, or lack of it on a curve's fracticality. Finally, this research brings into focus the strong problem solving capabilities, at the hands of cartographers, through the use of interactive computer graphics.

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May 12, 1989

Mr. Chuck Sultzman
U.S. Fish and Wildlife Service
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Dear Mr. Sultzman:

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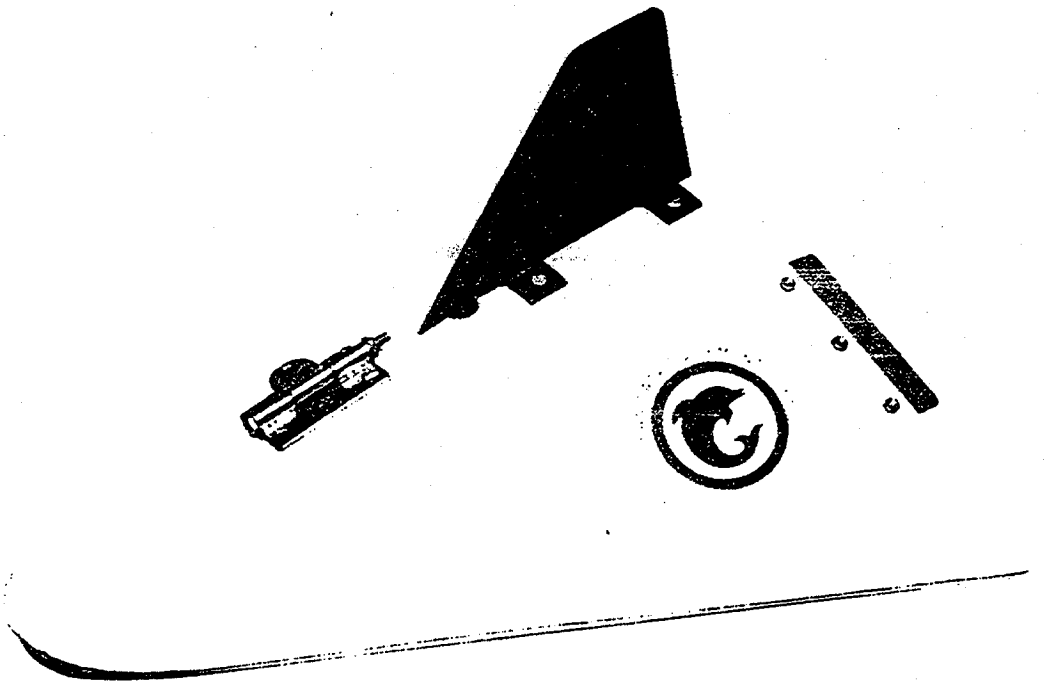

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